Non-trivial Solutions of the Bach Equation Exist

Hans - Jürgen Schmidt

Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10
D-14469 Potsdam, Germany, E-mail: hjschmi@rz.uni-potsdam.de

Abstract

We show that solutions of the Bach equation exist which are not conformal Einstein spaces.

In connection with fourth order gravitational field equations, cf. e.g. [1, 2] where the breaking of conformal invariance was discussed, the original Bach equation,

$$B_{ij} = 0\,, \tag{1}$$

enjoys current interest. Eq. (1) stems from a Lagrangian

$$L = \frac{1}{2}\sqrt{-g}C_{ijkl}C^{ijkl}\,,$$

and variation gives, cf. Bach [3],

$$\frac{1}{\sqrt{-g}}\delta L/\delta g^{ij} = B_{ij} = 2C^{a}{}_{ij}{}^{b}{}_{;ba} + C^{a}{}_{ij}{}^{b}R_{ba}\,.$$

An Einstein space,

$$R_{ij} = \lambda g_{ij}\,, \tag{2}$$

is always a solution of the BACH equation (1). But eq. (1) is conformally invariant, and therefore, each metric, which is conformally related to an Einstein space, fulfils eq. (1), too. We call such solutions trivial ones.

Now the question arises whether non-trivial solutions of the BACH equation (1) do or do not exist, and the present note will give an affirmative answer. As a by-product, some conditions will be given under which only trivial solutions exist. Observe that eq. (1) is conformally invariant whereas eq. (2) is not. Therefore, a simple counting of degrees of freedom does not suffice.

Because the full set of solutions of eq. (1) is not easy to describe, let us consider some homogeneous cosmological models. Of course, we have to consider anisotropic ones, because all Robertson-Walker models are trivial solutions of eq. (4). Here, we concentrate on the diagonal Bianchi type I models

$$ds^2 = dt^2 - a_i^2 dx^{i^2} \tag{3}$$

with Hubble parameters $h_i = a_i^{-1} da_i/dt$, $h = \Sigma h_i$ and anisotropy parameters $m_i = h_i - h/3$. The Einstein spaces of this kind are described in [4], for $\lambda = 0$ it is just the Kasner metric $a_i = t^{p_i}$, $\Sigma p_i = \Sigma p_i^2 = 1$. All these solutions have the property that the quotient of two anisotropy parameters, m_i/m_j, (which equals $(3p_i - 1)/(3p_j - 1)$ for the Kasner metric) is independent of t, and this property is a conformally invariant one. Furthermore, it holds: *A solution of eqs. (1), (3) is a trivial one, if and only if the quotients m_i/m_j are constants.*

Restricting now to axially symmetric Bianchi type I models, i.e., metric (3) with $h_1 = h_2$, the identity $\Sigma m_i = 0$ implies $m_1/m_2 = 1$, $m_3/m_1 = m_3/m_2 = -2$, i.e., *each axially symmetric Bianchi type I solution of eq. (1) is conformally related to an Einstein space.* (Analogously, all static spherically symmetric solutions of eq. (1) are trivial ones, cf. [5].)

Finally, the existence of a solution of eqs. (1), (3) with a non-constant m_1/m_2 will be shown. For the sake of simplicity we use the gauge condition

$h = 0$, which is possible because of the conformal invariance of eq. (1). Then the 00 component and the 11 component of eq. (1) are sufficient to determine the unknown functions h_1 and h_2; $h_3 = -h_1 - h_2$ follows from the gauge condition. Defining $r = (h_1^2 + h_1 h_2 + h_2^2)^{1/2}$ and $p = h_1/r$, eq. (1) is equivalent to the system

$$3d^2(pr)/dt^2 = 8pr^3 + 4c, \quad c = \text{const.}, \quad p^2 \leq 4/3\,, \tag{4}$$

$$9(dp/dt)^2 r^4 = [2rd^2r/dt^2 - (dr/dt)^2 - 4r^4](4r^2 - 3p^2r^2)\,. \tag{5}$$

As one can see, solutions with a non-constant p exist, i.e., m_1/m_2 is not constant for this case.

Result. *Each solution of the system (3), (4), (5) with $dp/dt \neq 0$ represents a non-trivial solution of the* BACH *equation (1).*

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